Exercise 12

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = \sin x + \cos x + \sinh x - 2\int_0^x \cosh(x-t)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx}f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\sin x + \cos x + \sinh x - 2\int_0^x \cosh(x-t)u(t)\,dt\right\}$$

$$U(s) = \mathcal{L}\{\sin x\} + \mathcal{L}\{\cos x\} + \mathcal{L}\{\sinh x\} - 2\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\}$$

$$= \mathcal{L}\{\sin x\} + \mathcal{L}\{\cos x\} + \mathcal{L}\{\sinh x\} - 2\mathcal{L}\{\cosh x\}U(s)$$

$$= \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2-1} - 2\left(\frac{s}{s^2-1}\right)U(s)$$

Solve for $U(s)$.

$$\left(1 + \frac{2s}{s^2-1}\right)U(s) = \frac{1+s}{s^2+1} + \frac{1}{s^2-1}$$

$$\frac{s^2+2s-1}{s^2-1}U(s) = \frac{s(s^2+2s-1)}{(s^2+1)(s^2-1)}$$

$$U(s) = \frac{s}{s^2+1}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$u(x) = \mathcal{L}^{-1}\{U(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \cos x$$